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Coupling Between Radial Betatron and Synchrotron Oscillations

by

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I. Introduction

There are several effects which arise from coupling between betatron and synchrotron oscillations. These can lead, in the presence of field bumps and gradient errors, to satellite stop bands near the integral and half-integral tunes. The purpose of this note is to review some of the effects and to estimate their importance for the NAL Booster and Main Ring.

The coupling effects which will be considered here are:

(a) coupling due to harmonics of the r-f accelerating force.

(b) coupling due to the dependence of the betatron frequency on longitudinal momentum. (This effect has been observed[1] and analyzed[2].)

(c) coupling due to the longitudinal dependence of the transverse space charge defocussing force. (This effect has been discussed by Möhl[3].)

II. Coupling Due to Accelerating Field Harmonics

A. Analysis

The (coupled) equations of motion for the linearized, smoothed radial betatron and synchrotron oscillations are (damping effects are neglected in this analysis):

$$\frac{d^2x}{dt^2} + (\nu^2-\gamma^2)\omega_o^2 x = \frac{r_o\omega_o\gamma^2}{h}\dot{\phi} \qquad (1a)$$

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$$\frac{d^2\phi}{dt^2} + \frac{h\omega_o}{r_o}\frac{dx}{dt} = -\frac{heV\cos\phi_o}{2\pi M\gamma^3 r_o^2}(\phi-\phi_o)\ [1 + v_m \cos m\omega_o t] \tag{1b}$$

The parameters are the usual ones[4], with

eV = maximum energy gain per turn

ω_o = rotation frequency

h = rf harmonic number

v_m = relative amplitude of the $m^{\underline{th}}$ harmonic (arising from sum of h + $m^{\underline{th}}$ and h - $m^{\underline{th}}$ spatial harmonic)

For h >> m, one can write

$$v_m \approx \frac{2}{J}\sum_{j=1}^{J} \cos m\theta_j \tag{2}$$

where the sum is taken over the J r-f accelerating gaps (assumed for convenience to be symmetric about $\theta = 0$).

It is possible to diagonalize the system of equations in (1) in the absence of v_m, leading to the usual separation into betatron and synchrotron oscillations. The v_m term then represents a coupling of the two oscillations. Starting with the Lagrangian,

$$\mathcal{L} = \frac{\gamma\dot{x}^2}{2} + \frac{r_o^2\gamma^3}{2h^2}\dot{\phi}^2 + \frac{r_o\omega_o\gamma^3}{h}\dot{\phi}x - \frac{\omega_o^2\gamma(\nu^2-\gamma^2)}{2}x^2$$
$$- \frac{eV\cos\phi_o}{2\pi Mh}\frac{(\phi-\phi_o)^2}{2}[1 + v_m \cos m\omega_o t] , \tag{3}$$

one obtains the equations of motion:

$$\frac{d^2x_\beta}{d\theta^2} + \nu^2 x_\beta = \nu_s^2 v_m x_\phi \cos m\theta, \quad \frac{d^2x_\phi}{d\theta^2} + \nu_s^2 x_\phi = \frac{\nu_s^2\gamma^2}{\nu^2-\gamma^2} v_m x_\beta \cos m\theta \tag{4}$$

where x_β and x_ϕ are the radial displacements in the two modes, ν_s is the phase oscillation frequency, and only terms of zero or first order in $\frac{eV}{Mc^2\beta^2}$ have been retained.

These equations of motion show stop bands at:

$$\nu = m - \nu_s \text{ for } \nu^2 > \gamma^2 \text{ (below transition)}$$

and (5)

$$\nu = m + \nu_s \text{ for } \gamma^2 > \nu^2 \text{ (above transition) .}$$

The total width of these stop bands is:

$$\Delta\nu_{SB} = v_m\left(\frac{\nu_s}{\nu}\right)^{3/2} \frac{\gamma\nu}{|\nu^2-\gamma^2|^{1/2}} \tag{6}$$

and the e-folding rate at the center of the bands is:

$$\frac{1}{\tau} = \frac{\omega_o}{4}\Delta\nu_{SB} \quad . \tag{7}$$

For the combination of signs opposite to those of (5), there is no stop-band, but in either case there is a modulation of the amplitudes due to the coupling, even away from resonance. The peak amplitude may be estimated from the invariants formed from the Hamiltonian, or by using phase amplitude methods in (4), leading to:

$$\delta A_\phi = \frac{v_m}{4}\frac{\nu_s}{\Delta\nu}\frac{\gamma^2}{\nu^2-\gamma^2}A_\beta \tag{8a}$$

$$\delta A_\beta \sim \frac{v_m}{4\nu}\frac{\nu_s^2}{\Delta\nu}A_\phi \tag{8b}$$

where

$$\Delta\nu = |m - \nu \pm \nu_s| \quad . \tag{9}$$

Here δA_β and δA_ϕ are the increases in radial amplitudes of the betatron and phase oscillations (maximum values of x_β and x_ϕ). The amplitudes also satisfy the invariant

$$A_\beta^{\ 2} \pm \frac{\nu_s |\nu^2 - \gamma^2|}{\gamma^2 \nu} A_\phi^{\ 2} = \text{const.} \tag{10}$$

The oscillation growths in Eq. (8) are proportional to v_m, the relative amplitude of the $m^{\underline{th}}$ spatial harmonic of the r-f, as defined in (2). Regular spacing and phasing of r-f cavities in azimuth may lead to the absence of particular harmonics. Indeed, if it proves necessary to eliminate a troublesome harmonic, this may be done by appropriate choice of the azimuth of one or more r-f cavities.

B. Numerical Values (taken from NAL Design Report, July 1968)

1. Booster

With $\nu_x = 6.7$

$\nu_s = 0.08$

$\gamma = 1.2$

$m = 7$

$\Delta\nu = 0.22$

$$v_7 \simeq \left| \frac{\cos 210^\circ + \cos 315^\circ + \cos 420^\circ + \cos 525^\circ}{2} \right| \simeq .31$$

we find:

$$\Delta\nu_{SB} = 10^{-4}$$

$$\delta A_\phi = 10^{-3} A_\beta$$

$$\delta A_\beta = 3 \times 10^{-4} A_\phi$$

2. Main Ring

With $\nu_x = 20.25$

$\nu_s = 0.02$

$\gamma = 10$

$m = 20$

$v_m \simeq 2$

$\Delta\nu = 0.23$

we find:

$$\Delta\nu_{SB} = 7 \times 10^{-4}$$

$$\delta A_\phi = 10^{-2} A_\beta$$

$$\delta A_\beta = 4 \times 10^{-5} A_\phi$$

The effect of coupling due to harmonics of the accelerating field is clearly quite small.

III. Coupling Due to Dependence of Betatron Frequency on Momentum

A. Analysis for Field Bumps

The coupled equations which contain a variation of betatron frequency with longitudinal momentum can also be derived from the Hamiltonian, provided one adds a term of the form $x_\beta^2 x_\phi$. Computation of the side bands then is governed by the effect of a field bump (or gradient error) in the presence of the coupling term.

Although separate differential equations are obtained for x_β and x_ϕ, the effect of x_β on x_ϕ is negligible, so that the zero-order solution for x_ϕ may be used in the equation for x_β. The equation for x_β with a field bump is then:

$$\frac{d^2x}{d\theta^2} + x[\nu^2 + 2\nu\Delta\nu \cos(\nu_s\theta + \psi)] = b \cos m\theta \qquad (11)$$

where $\Delta\nu$ is the amplitude of variation of betatron frequencies, $\nu_s\theta + \psi$ is the phase of the phase oscillation, and b is the amplitude of the $m^{\underline{th}}$ Fourier component of the field error. A similar equation applies to the vertical motion, with the appropriate b and $\Delta\nu$.

As has been previously shown[2], the solution for the homogeneous part of (11) for small $\frac{\Delta\nu}{\nu}$ is:

$$x \simeq e^{\pm i\int d\theta[\nu + \Delta\nu \cos(\nu_s\theta + \psi)]} = e^{\pm i[\nu\theta + \frac{\Delta\nu}{\nu_s}\sin(\nu_s\theta + \psi)]}$$

$$= \sum_{n=-\infty}^{\infty} J_n\left(\frac{\Delta\nu}{\nu_s}\right) e^{\pm i[(\nu + n\nu_s)\theta + \psi]} \; . \tag{12}$$

One can form a Green's function and integrate (11). The amplitude of the closed orbit distortion then becomes

$$A \simeq \frac{b}{2\nu}\left|\sum_n \frac{J_n(\Delta\nu/\nu_s)}{m-\nu-n\nu_s}\, e^{-in(\nu_s\theta + \psi)}\right| \tag{13}$$

If one value of n dominates, (13) becomes

$$A \simeq \frac{J_n(\Delta\nu/\nu_s)}{2\nu(m-\nu-n\nu_s)} \tag{14}$$

Obviously for appropriate values of m, ν, n, and ν_s, the denominator in (14) may vanish, leading to an unacceptable growth in the oscillation amplitude. However, ν_s changes during acceleration, and will not sit indefinitely on a resonance.

For those values of n for which $m-\nu-n\nu_s$ does not vanish during acceleration, (14) may be used to estimate the closed orbit amplitude. For those values of n for which $m-\nu-n\nu_s$ does vanish during acceleration, one can calculate the free oscillation amplitude induced in passing through the resonance in terms of the rate of change of ν_s at resonance.[2] The result is:

$$\mathcal{A} \simeq \frac{\pi b}{\nu}\left[\frac{2\pi n}{\omega_o}\frac{d\nu_s}{dt}\right]^{-1/2} J_n\left(\frac{\Delta\nu}{\nu_s}\right) \tag{15}$$

where ν_s and $d\nu_s/dt$ are to be evaluated where $m-\nu-n\nu_s$ vanishes.

The results in (14) and (15) can conveniently be expressed in terms of A_m, the amplitude of the closed orbit distortion due to the $m^{\underline{th}}$ harmonic of the error in the absence of coupling:

$$A_m \sim \frac{b}{2\nu|m-\nu|} \quad . \tag{16}$$

Thus,

$$\frac{A}{A_m} = \frac{m-\nu}{m-\nu-n\nu_s} J_n\left(\frac{\Delta\nu}{\nu_s}\right) \tag{17}$$

and

$$\frac{\mathcal{Q}}{A_m} = 2\pi(m-\nu)\left[\frac{2\pi n}{\omega_o}\frac{d\nu_s}{dt}\right]^{-1/2} J_n\left(\frac{\Delta\nu}{\nu_s}\right) \tag{18}$$

It is instructive to examine (17) in the limit of small ν_s. For large argument, the Bessel function in (17) peaks at the order

$$n \sim \Delta\nu/\nu_s$$

after which it falls rapidly to zero as the order n increases. This implies the necessity for avoiding the range

$$\nu = m \pm \Delta\nu \tag{19}$$

which is the expected result for a "tune" which swings between $\nu - \Delta\nu$ and $\nu + \Delta\nu$. This is the adiabatic result to which one is led by following the movement of the operating point during acceleration.

For $\Delta\nu/\nu_s$ of order 1 or less however, (17) and (18) may be appreciable for values of n as high as 3 or 4. The sidebands in this case may extend beyond the limits in (24), thus more seriously restricting the extent to which the operating point may wander in ν_x.

B. Numerical Examples

In the main ring, ν_s is sufficiently small that expression (19) adequately describes the restriction on ν. In the booster, however, the side-band effects could be significant. To give some idea of the numbers involved, we take as an example,

$\nu_x = 6.7, \quad \nu_z = 6.8$

$m = 7$

$\nu_s = 0.08$

Table I gives the contributions to the non-resonant closed orbit deviation $(\frac{A_x}{A_{x7}}, \frac{A_z}{A_{z7}})$ for a range of $\Delta\nu$.

Table I

$\Delta\nu \diagdown n$	0	1	2	3	4
0	1, 1	0, 0	0,0	0,0	0,0
.02	.99, .99	.17, .28	---	---	---
.04	.96, .96	.33, .55	.07, .16	---	---
.06	.93, .93	.47, .78	.14, .33	.04, .04	---
.08	.87, .87	.59, .98	.25, .59	.10, .10	.04, .01

These numbers are not alarming in themselves, indicating at most a factor of two in closed orbit amplitude for $\Delta\nu \sim 0.1$, which is considerably larger than anticipated in the design. However, the numbers are very sensitive to the choice of parameters, so that it would be advisable to plan on keeping the tunes at or below 6.8 until ν_s is well below 0.08 and on controlling $\Delta\nu$, by sextupole trimming, if necessary. As an indication of what can happen in passage through resonance, we take $\nu = 6.8$, $n = 3$, $\frac{\Delta\nu}{\nu_s} = 1$, and $\frac{d\nu_s}{dt} = 15 \text{ sec}^{-1}$ (from the initial slope in Fig. 9-8 of the Design Report). Then, from equation (18), $\frac{\mathcal{a}}{A_7} \sim 2.5$ which should be doubled for the subsequent passage through resonance as ν_s decreases. That is to say, a coherent oscillation would be induced of amplitude five times that of the normal closed orbit deviation. Fortunately, for small x, $J_n(x) \sim \frac{1}{n!}(\frac{x}{2})^n$, so that the effect decreases rapidly with decreasing $\Delta\nu$ and increasing $n(= \frac{m-\nu}{\nu_s})$; again the solution would be to depress $\Delta\nu$ during the critical period.

C. Analysis for Gradient Errors

The relevant equation is (11) with a right side proportional to x or z:

$$\frac{d^2x}{d\theta^2} + x[\nu^2 + 2\nu\Delta\nu \cos(\nu_s\theta + \psi)] = xk \cos m\theta \tag{20}$$

where k is the amplitude of the $m^{\underline{th}}$ harmonic of the gradient error. The analysis parallels that leading to (13). In this case one finds a fractional amplitude growth:

$$\frac{\delta A}{A} = \frac{k}{4\nu}\left|\Sigma \frac{J_n(2\Delta\nu/\nu_s)}{m-2\nu-n\nu_s} e^{-in(\nu_s\theta + \psi)}\right| . \tag{21}$$

For passage through a single resonance at $v = (m - n\Delta\nu_s)/2$ one obtains:

$$\frac{\delta A}{A} \approx \frac{\pi k}{2\nu} \left[\frac{2\pi n}{\omega o} \frac{d\nu_s}{dt}\right]^{-1/2} J_n(2\Delta\nu/\nu_s). \tag{22}$$

It is convenient to replace k by the stop-band width it would produce at the nearest half-integer tune:

$$\Delta_o = \frac{k}{2\nu} . \tag{23}$$

Then (21) and (22) become, respectively,

$$\frac{\delta A}{A} = \frac{\Delta_o}{2}\left|\frac{J_n(\frac{2\Delta\nu}{\nu_s})}{m-2\nu-n\nu_s}\right|, \text{ (for a single n)} \tag{24}$$

and

$$\frac{\delta A}{A} = \pi\Delta_o\left[\frac{2\pi n}{\omega_o} \frac{d\nu_s}{dt}\right]^{-1/2} J_n\left(\frac{2\Delta\nu}{\nu_s}\right) . \tag{25}$$

Since Δ_o is expected to be $\sim 10^{-2}$, the non-resonant beating effect (24), is quite small. Also, for $\Delta\nu \lesssim \nu_s$, (25) yields increases less serious than those of the preceding section.

IV. Coupling Due to Space Charge

The presence of space charge adds to the complication in the coupling. We shall consider here the effect treated by Möhl[3], which takes into account the variation of transverse frequency with longitudinal position in the bunch, due to the variation of the transverse force constant with longitudinal position.

If one takes a parabolic variation in the force constant (corresponding roughly to a parabolic variation with azimuth of charge density), one obtains the equation,

$$\frac{d^2x}{d\theta^2} + \nu^2 x + 2\nu\Delta\nu_{sc}(x - \bar{x})\left(1 - \frac{\phi^2}{\phi_{max}^2}\right) = \begin{cases} b \cos m\theta \\ \\ xk \cos m\theta \end{cases} \tag{26}$$

in the presence of field bumps or gradient errors. Here $\bar{x}$ is the average radial displacement of the bunch at a given longitudinal phase, $2\phi_{max}$ is the phase length of the bunch, and $\Delta\nu_{sc}$ is the change in tune at the center of the bunch.

It is not difficult to show that, if image forces are neglected, the field bump causes the beam to acquire a fixed orbit distortion coherently, and that the space charge term affects only the relative motion of the individual particles, not their average. The gradient errors lead to no displacement of the bunch center, but to an increase in diameter if resonance occurs. An analysis similar to that of Orlov[2] leads Möhl to the result in (25) with the replacements

$$\left.\begin{aligned} &\nu \rightarrow \nu - \Delta\nu_{sc}\left(1 - \frac{\phi_1^2}{2\phi_m^2}\right) \\ &\Delta\nu \rightarrow \Delta\nu_{sc}\,\frac{\phi_1^2}{2\phi_m^2} \\ &\nu_s \rightarrow 2\nu_s \end{aligned}\right\} \tag{27}$$

where ϕ_1 is the amplitude of the synchrotron oscillation for the particular collection of particles being considered. One then obtains:

$$\frac{\delta A}{A} \approx \pi\Delta_o\left[\frac{4\pi n}{\omega_o}\frac{d\nu_s}{dt}\right]^{-1/2} J_n\left(\frac{\Delta\nu_{sc}}{2\nu_s}\frac{\phi_1^2}{\phi_m^2}\right) \quad , \tag{28}$$

where the resonance occurs for

$$\left|\frac{m}{2} - \nu + \Delta\nu_{sc}\left(1 - \frac{\phi_1^2}{2\phi_m^2}\right)\right| = n\nu_s \quad . \tag{29}$$

In order to determine the order of magnitude of (28), we shall consider the extreme particles in the synchrotron oscillations of the booster ($\phi_1 = \phi_m$) and a value of $\Delta\nu_{sc} = .2$. The resonance condition then becomes, with m = 13, $\nu = 6.7$,

$$n\nu_s = .1$$

which can occur for n = 1, $\nu_s = 0.1$ and/or n = 2, $\nu_s = 0.05$. From (28), with $\frac{d\nu_s}{dt} = 15\ \mathrm{sec}^{-1}$, one finds

$$\left(\frac{\delta A}{A}\right)_1 = 150\ \Delta_o \tag{30a}$$

$$\left(\frac{\delta A}{A}\right)_2 = 85\ \Delta_o \tag{30b}$$

For $\Delta_o \sim 10^{-2}$, (30) looks alarmingly large. Moreover, the booster parameters are such that the beam would be right on the n = 2, $\nu_s = 0.05$ stop-band at injection, which would lead to a still larger figure. However, the driving force is so weak that non-linear effects cannot be neglected. If, for example, the charge distribution in the beam is taken to be parabolic in the transverse dimensions, rather than uniform, the resulting cubic terms in the equations of motion would limit amplitude growths to something less than the envelope defined by the largest amplitude oscillations. Furthermore, on the basis of linear theory, the widths of the satellite stop-bands are given by:

$$\Delta_n = \Delta_o J_n \left(\frac{\Delta\nu_{sc}}{2\nu_s} \frac{\phi_1^{\,2}}{\phi_m^{\,2}} \right) \quad , \tag{31}$$

which are smaller than Δ_o. Therefore, if the gradient errors can be trimmed out to the point where a low intensity beam can ride on half-integral tune, as has been accomplished in existing machines, the satellite stop-bands should not be troublesome. A more detailed study of these speculative remarks should probably be made.

References

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